Examples of M5-Brane Elliptic Genera

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Abstract

We determine the modified elliptic genus of an M5-brane wrapped on various one modulus Calabi-Yau spaces, using modular invariance together with some known Gopakumar-Vafa invariants of small degrees. As a bonus, we find nontrivial relations among Gopakumar-Vafa invariants of different degrees and genera from modular invariance.

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1. Introduction

The modified elliptic genus of an M5-brane wrapped on a Calabi-Yau space counts the degeneracies of D4-D2-D0 BPS black holes [1,2,3]. This partition function is modular invariant and can be determined entirely by the knowledge of the degeneracies of a finite number of states [4,5,6,7,8,3]. See also [9,10,11]. This was applied in [3] to an M5-brane wrapped on the hyperplane section of the quintic threefold. In this note, we extend the result of [3] to some other Calabi-Yau spaces: the sextic, octic, dectic in weighted projective spaces, as well as the bicubic in $\mathbf{P}^5$.

The modified elliptic genus of an M5-brane wrapped on an ample divisor P in Calabi-Yau space X takes the form

$$Z_{X,P}(\tau,\bar\tau,y^A) = \sum_{\delta\in\Lambda^*/\Lambda} Z_\delta(\tau)\Theta_{\Lambda+\delta}(\tau,\bar\tau,y^A) \tag{1.1}$$

where $\Lambda \subset H^2(P,\mathbf{Z})$ is the image of

$$\iota : H^2(X,\mathbf{Z}) \hookrightarrow H^2(P,\mathbf{Z})$$

$\Theta_{\Lambda+\delta}$ is the theta functions of the shifted lattice $\Lambda+\delta$,[1]

$$\Theta_{\Lambda+\delta}(\tau,\bar\tau,y^A) = \sum_{\vec q\in\Lambda+\delta+\frac{J}{2}} (-)^{J\cdot q}\exp\left[-\pi i\tau\vec q^{\,2} + \pi i(\tau-\bar\tau)\frac{(J\cdot q)^2}{J\cdot J} + 2\pi i y\cdot q\right] \tag{1.2}$$

[1] There is an additional half integral shift by $J/2$ due to a well known anomaly [12,13].

where J is the canonical class of P. $Z_\delta(\tau)$ are a set of holomorphic modular vectors.[2] $Z_{X,P}$ is expected to be a Jacobi form of weight $(-\frac{3}{2}, \frac{1}{2})$.

Our approach, as in [3], is to determine $Z_{X,P}$ from the polar terms in the q-expansion of Z_δ's. The latter involves the degeneracy of BPS D4-D2-D0 bound states with small charges, and can be determined from geometric reasoning. In [3], the geometric counting was "naive" in that the authors did not take into account singularities in the classical moduli space of the D4-D2-D0 bound states, which need to be resolved. There one needed to invoke arguments based on the holographic dual of the M5-brane $(0,4)$ CFT to get the precise counting. In this note we proposed a more refined counting solely based on the geometry, and we will see that it gives precisely the correct countings that are consistent with the constraints imposed by modular invariance.

We will count D4-D2-D0 bound states with small charges by quantizing their classical moduli space. The classical supersymmetric configuration of D4-D2-D0 system involves a hypersurface P (hyperplane in our examples), with $U(1)$ fluxes represented by a type $(1,1)$ harmonic form F, together with n point-like instantons (D0-branes). Up to the shift by $J/2$, F represents an integral class in $H^2(P, \mathbf{Z})$, and can be represented by an integral linear combination of holomorphic curves C_i in P. It is most convenient to think of C_i's as curves in X that coincide with P. Note that two curves can be homologous in X but not homologous as classes in P. We will mostly think of the simple case when C_i's are rigid curves. In general they are counted by Gopakumar-Vafa invariants [15,16,17].[3] One can then think of (a component of) the classical moduli space as the space $\mathcal{M}$ of hyperplane P that passes through a set of given curves C_i as well as n points in X.

With C_i's rigid and fixed, $\mathcal{M}$ is essentially a projective space fibered over the space of n points in X. The index that counts BPS states is given by the Euler characteristic (in a suitable sense) of $\mathcal{M}$, which is easy to evaluate on the smooth components of $\mathcal{M}$. The naive description of $\mathcal{M}$ based on classical geometry has a lot of singularities. For example, wherever some of n points coincide with one another, or coinciding with one of the curves C_i, the dimension of the fiber projective space jumps and $\mathcal{M}$ is singular. Physically, such singularities can often be resolved by the nonabelian degrees of freedom of the D-branes.

[2] When P is not ample, there can be a holomorphic anomaly in the Z_δ's [14]. This subtlety does not appear in the examples we will be considering, and will be ignored in this note.

[3] When the curves have moduli, it is a priori not obvious that Gopakumar-Vafa invariants are relevant for our counting. However, holography suggests that this should be the case [3].

We expect $\mathcal{M}$ to be fibered over a resolved space of n points in X (possibly the Hilbert scheme).

For example, when we have two points p_1, p_2 colliding in X, it is straightforward to resolve the moduli space, replacing the locus where p_1, p_2 coincide by the space of directions along which p_2 can approach p_1, namely a $\mathbf{P}^2$. Similarly, if a point p collides with a curve C, we will replace the locus in the moduli space where p lies on C by the space of possible directions p can approach C (a $\mathbf{P}^1$ worth of them) fibered over C. When three points collide, we can replace the locus where the three points coincide by the space of planes spanned by three points infinitesimally close to one another (a $\mathbf{P}^2$ worth of them). The resolution of the moduli space is not so straightforward when more than three points collide. It can presumably be understood in terms of the nonabelian dynamics of the D-branes. Fortunately we will not need them in the examples considered in this note.

In the following section we will compute the modified elliptic genus for an M5-brane wrapped on the hyperplane section in the quintic in $\mathbf{P}^4$, sextic in $\mathbf{WP}_{2,1,1,1,1}$, octic in $\mathbf{WP}_{4,1,1,1,1}$, dectic in $\mathbf{WP}_{5,2,1,1,1}$, and the bicubic in $\mathbf{P}^5$. We will make use of the Gromov-Witten and Gopakumar-Vafa invariants computed in [18,17] (more complete results can be found in [19]). The details of the modular vectors involved are described in the appendix.

2. The M5-brane elliptic genus on a number of Calabi-Yau spaces

2.1. The quintic in $\mathbf{P}^4$*, revisited*

In this subsection we recall the result of [3], but will recount the degeneracies of BPS states of small charges from a refined geometric picture. The modified elliptic genus takes the form

$$Z_{X_5}(\tau,\bar{\tau},y) = \sum_{i=0}^{4} Z_i(\tau)\Theta_i^{(5)}(\bar{\tau},y) \tag{2.1}$$

where

$$\Theta_k^{(m)}(\tau,y) \equiv \sum_{n\in\mathbf{Z}+\frac{1}{2}+\frac{k}{m}} (-)^{mn} q^{\frac{m}{2}n^2} e^{2\pi i y m n} \tag{2.2}$$

and the Z_i's are given by

$$\begin{aligned}
Z_0(q) &= q^{-\frac{55}{24}}(5 - 800q + 58500q^2 + 5817125q^3 + 75474060100q^4 + 28096675153255q^5 + \cdots) \\
Z_1(q) &= Z_4(q) = q^{-\frac{83}{120}}(8625 - 1138500q + 3777474000q^2 + 3102750380125q^3 + \cdots) \\
Z_2(q) &= Z_3(q) = q^{\frac{13}{120}}(-1218500 + 441969250q + 953712511250q^2 + 217571250023750q^3 + \cdots)
\end{aligned} \tag{2.3}$$

As a nontrivial check, the number of D4 bound to 2 D0-branes can be counted by considering a hyperplane that passes through two points, say p_1, p_2. When p_1 and p_2 are distinct, there is a $\mathbf{P}^2$ worth of hyperplanes that pass through both points. When p_1, p_2 collide, we need to resolve the moduli space and take into account the directions p_2 can approache p_1. This amounts to replace the locus in the moduli space where the two points collide by a $\mathbf{P}^2$. In this case, we shall require not only $p_1 = p_2$ lie in the hyperplane, but the vector determined by the direction along which p_2 approaches p_1 lie in the hyperplane as well. This again determines a $\mathbf{P}^2$ worth of hyperplanes. The counting is

$$(-200) \cdot (-201)/2 \cdot \chi(\mathbf{P}^2) + (-200) \cdot \chi(\mathbf{P}^2) \cdot \chi(\mathbf{P}^2) = 58500,$$

which indeed agrees with the corresponding coefficient in Z_0, predicted by modular invariance.

Next let us consider a D4 with one unit of flux, and bound to one extra D0. This is counted by a hyperplane that passes through a degree 1 rational curve C_1, as well as an extra point p. When p does not lie on C_1, p and C_1 determine a $\mathbf{P}^1$ worth of hyperplanes. When p collides with C_1, the moduli space is resolved so that it contains the space of directions along which p can approach C_1 at any given point, which is another $\mathbf{P}^1$. So that counting is

$$2875 \cdot (-200 - \chi(C_1)) \cdot \chi(\mathbf{P}^1) + 2875 \cdot \chi(C_1) \cdot \chi(\mathbf{P}^1) \cdot \chi(\mathbf{P}^1) = -1138500,$$

precisely agreeing with the corresponding coefficient in Z_1.

Now consider a D4 with two units of fluxes and D0-brane charge one more than the minimal value (the second coefficient in the q-expansion of Z_2). The counting receives three contributions: a hyperplane that passes through a degree 2 rational curve C_2 and a point p, with the flux being $F = C_2$; a hyperplane that passes through two distinct degree 1 rational curves C_1 and C_1', with the flux being $F = C_1 + C_1'$; or a hyperplane that passes through a degree 3 rational curve C_3, with the flux being $F = J - C_3$. In the first case, we again need to resolve the locus of the moduli space where p collides with C_2, as before. There is also an extra minus sign one needs to take into account as in [3].[4] Using the well known Gromov-Witten invariants of degree 1,2,3, the counting is

$$(-609250) \cdot \left[-200 - \chi(C_2) + \chi(C_2) \cdot \chi(\mathbf{P}^1)\right] + \binom{2875}{2} + 317206375 = 441969250,$$

[4] We do not know how to understand this directly from quantizing the classical moduli space. This is not a contradiction since disconnected branches of the moduli space can contribute with different signs. This sign was determined in [3] from the fermion number of the wrapped M2-brane in the holographic dual.

which again agrees with the prediction from modular invariance.

A more difficult example is a D4 bound to 3 D0's. There are essentially two kinds of contributions: a hyperplane that passes through two different degree 1 rational curves C_1 and C_1', with the flux being $F = C_1 - C_1'$; or a hyperplane that passes through three points p_1, p_2, p_3. The contribution from the first case is straightforward: C_1 and C_1' complete determines a hyperplane. The second case is more subtle due to the different configurations of the three points. Naively, there are five different situations one must consider:

(a) p_1, p_2, p_3 are distinct and are not aligned in the ambient $\mathbf{P}^4$. The three points determine a $\mathbf{P}^1$ worth of hyperplanes.

(b) p_1, p_2, p_3 are distinct and lie on a line L in $\mathbf{P}^4$ (which intersects the quintic at five points). L determines a $\mathbf{P}^2$ (as opposed to a $\mathbf{P}^1$) worth of hyperplanes.

(c) $p_1 = p_2 \neq p_3$. When resolving the moduli space by taking into account of the direction $\overline{p_1 p_2}$, p_3 does not lie on the line determined by $\overline{p_1 p_2}$ in the ambient $\mathbf{P}^4$.

(d) $p_1 = p_2 \neq p_3$. p_3 is one of the remaining 3 intersections of the line determined by $\overline{p_1 p_2}$ with the quintic in the $\mathbf{P}^4$.

(e) $p_1 = p_2 = p_3$. Resolving the moduli space replaces the point by a $\mathbf{P}^2$ worth of planes spanned by three close by points. Each such plane determines a $\mathbf{P}^1$ worth of hyperplanes.

Putting these together, we get the counting

$$
\begin{aligned}
&2875 \cdot 2874 + \frac{(-200)\cdot(-201)\cdot(-200-5)}{6}\chi(\mathbf{P}^1) + \frac{(-200)\cdot(-201)\cdot 3}{6}\chi(\mathbf{P}^2) \\
&+ (-200)\cdot(-200-4)\cdot\chi(\mathbf{P}^2)\cdot\chi(\mathbf{P}^1) + (-200)\cdot 3\cdot\chi(\mathbf{P}^2)\cdot\chi(\mathbf{P}^2) + (-200)\cdot\chi(\mathbf{P}^2)\cdot\chi(\mathbf{P}^1) \\
&= 5814250 = 5817125 - 2875.
\end{aligned}
$$

It is striking yet puzzling that the result differs from the prediction from modular invariance, 5817125, by -2875 (recall that 2875 is the number of degree 1 rational curves in the quintic). In the above counting we have ignored the more complicated situation where the points p_1, p_2, p_3 lie on a degree 1 curve C_1 (as opposed to a generic line L). The corrections one obtain by taking into account such configurations will presumably be a multiple of 2875. We do not understand why the multiplicity is "1", which we will leave to future investigation.

In summary, we found remarkable agreement of the modified elliptic genus with the proposed geometric counting by resolving the singularities of the moduli space.

2.2. Degree 6 hypersurface in $\mathbf{WP}_{(2,1,1,1,1)}$

The Calabi-Yau 3-fold X_6 is defined as the hypersurface

$$x_1^3 + x_1 f_4(x_2, x_3, x_4, x_5) + f_6(x_2, x_3, x_4, x_5) = 0 \tag{2.4}$$

in the weighted projective space $\mathbf{WP}_{(2,1,1,1,1)}$, where f_4 and f_6 are polynomials of homogeneous degree 4 and 6 in $x_2, \cdots, x_5$. We will assume that f_4, f_6 are generic and X_6 is smooth. The choice of complex structure is not essential for our purpose. X_6 has $h^{1,1} = 1$, $h^{2,1} = 103$, $\chi = -204$, $c_2 = 14h$, h being generator of $H^4(X_6, \mathbf{Z})$. The hyperplane section P has $6D = P \cdot P \cdot P = 3$, $c_2 \cdot P = 42$. The M5-brane $(0,4)$ CFT has left and right central charges

$$c_L = 6D + c_2 \cdot P = 45, \quad c_R = 6D + \tfrac{1}{2} c_2 \cdot P = 24.$$

The modified elliptic genus takes the form

$$Z_{X_6}(\tau, \bar{\tau}, y) = \sum_{i=0}^{2} Z_i(\tau) \Theta_i^{(3)}(\bar{\tau}, y) \tag{2.5}$$

where the $\Theta_i^{(3)}$'s are defined as in (2.2), and $Z_1 = Z_2$. A direct counting from geometry gives the polar terms

$$\begin{aligned} Z_0(\tau) &= q^{-\frac{45}{24}}(4 + 3 \cdot (-204) q + \cdots) \\ Z_1(\tau) &= q^{-\frac{45}{24} - \frac{1}{3} + 2}(2 \cdot 7884 + \cdots) \end{aligned} \tag{2.6}$$

where $\cdots$ are non-polar terms, of higher orders in q. This determines the modified elliptic genus by modular invariance. We will leave the details of the modular forms to the appendix, and write the first few terms in the q-expansion here

$$\begin{aligned} Z_0(\tau) &= q^{-\frac{45}{24}}(4 - 612q + 40392q^2 - 146464860q^3 - 66864926808q^4 - 8105177463840q^5 \\ &\quad - 503852503057596q^6 - 20190917119833144q^7 - 587565090039987648q^8 + \cdots), \\ Z_1(\tau) &= Z_2(\tau) = q^{-\frac{5}{24}}(15768 - 7621020q - 10739279916q^2 - 1794352963536q^3 \\ &\quad - 134622976939812q^4 - 6141990299963544q^5 - 196926747589177416q^6 + \cdots). \end{aligned} \tag{2.7}$$

Let us make a few checks. The number of D4 bound to 2 D0's can be counted directly from the geometry. Naively, by resolving the moduli space of a hyperplane passing through two points p_1, p_2 as before we get

$$(-204)(-205)/2 \cdot \chi(\mathbf{P}^1) + (-204) \cdot \chi(\mathbf{P}^2) \cdot \chi(\mathbf{P}^1) = 40596$$

which differs from the expected answer 40392 by 204. The reason for this discrepancy is a simple geometric fact: hyperplane sections of the sextic are defined by linear equations in the four degree 1 variables $x_2, \cdots, x_5$ only. Given a point p_1, all hyperplanes through p_2 will also pass through two other points in the sextic with the same $x_2, \cdots, x_5$ coordinates but with different x_1 coordinates. If p_2 is one of these two points, it will not constrain the hyperplane any further, and hence there is a $\mathbf{P}^2$, instead of a $\mathbf{P}^1$, worth of hyperplanes through p_1, p_2. This gives a correction $(-204) \cdot 2/2$ to the degeneracy. In the end we get $40596 - 204 = 40392$ which precisely agrees with (2.7).

The number of D4 bound to 1 D2 and 1 D0 can be counted directly:

$$-6028452 + (-204 - 2) \cdot 7884 + 2 \cdot 2 \cdot 7884 = -7621020$$

which again exactly matches the predicted answer in $Z_1(\tau)$.

2.3. Degree 8 hypersurface in $\mathbf{WP}_{(4,1,1,1,1)}$

The Calabi-Yau 3-fold X_8 is defined as the hypersurface

$$x_1^2 + f_8(x_2, x_3, x_4, x_5) = 0 \tag{2.8}$$

in the weighted projective space $\mathbf{WP}_{(4,1,1,1,1)}$, with f_8 a polynomial of homogeneous degree 8. X_8 has $h^{1,1} = 1$, $h^{2,1} = 149$, $\chi = -296$, $c_2 = 22h$. The hyperplane section P has $6D = P \cdot P \cdot P = 2$, $c_2 \cdot P = 44$. The M5-brane $(0,4)$ CFT has central charges

$$c_L = 6D + c_2 \cdot P = 46, \quad c_R = 6D + \tfrac{1}{2} c_2 \cdot P = 24.$$

The modified elliptic genus takes the form

$$Z_{X_8}(\tau, \bar\tau, y) = Z_0(\tau)\theta_2(2\bar\tau, 2y) - Z_1(\tau)\theta_3(2\bar\tau, 2y) \tag{2.9}$$

Direct counting from geometry gives the polar terms of $Z_{0,1}(\tau)$,

$$\begin{aligned} Z_0(\tau) &= q^{-\frac{46}{24}}(4 + 3 \cdot (-296)q + \cdots) \\ Z_1(\tau) &= q^{-\frac{46}{2} - \frac{1}{4} + 2}(2 \cdot 29504 + \cdots) \end{aligned} \tag{2.10}$$

These determine the modified elliptic genus completely. We will leave the details of the modular forms to the appendix, and write the first few terms in the q-expansion here

$$\begin{aligned} Z_0(\tau) &= q^{-\frac{46}{12}}(4 - 888q + 86140q^2 - 132940136q^3 - 86849300500q^4 \\ &\quad - 11756367847000q^5 - 787670811260144q^6 - 33531427162546608q^7 + \cdots) \\ Z_1(\tau) &= q^{-\frac{1}{6}}(59008 - 8615168q - 21430302976q^2 - 3736977423872q^3 \\ &\quad - 289181439668352q^4 - 13588569634434304q^5 - 448400041603851008q^6 + \cdots) \end{aligned} \tag{2.11}$$

Let us make a few checks. The direct counting of D4 bound to D2 and a D0 gives the $q^{\frac{5}{6}}$ coefficient of $Z_1(q)$

$$29504 \cdot (-296 - 2) + 2 \cdot \chi(\mathbf{P}^2) 29504 = -8615168$$

which exactly matches the prediction from modularity. A direct counting of D4 bound to 2 D0 gives

$$2(-296) \cdot (-297)/2 + 2\chi(\mathbf{P}^2)(-296) = 86136$$

which differs from the expected answer 86140 in $Z_0(q)$ by 4. Similar to the case of the sextic, we expect a correction due to the fact that a hyperplane though p_1 also necessarily passes through one other point p_2 with the same $x_2, \cdots, x_5$ coordinates as p_1. This would give a correction $(-296) \cdot 1/2 = -148$ to the degeneracy. This brings the discrepancy with the expected answer from (2.11) to 152. This does not necessarily imply a failure of the geometric counting, since there are potentially holomorphic curves that can contribute to the number of BPS states with the same charges. A degree $d = 2m$ genus g curve C in P has self-intersection $C \cdot C = 2g - 2 - 2m$ and turning on the flux $F = C - mJ$ would induce D0 charge $-\frac{(C-mJ)^2}{2} = m^2 + m + 1 - g$. For example, any $d = 2, g = 1$ or $d = 4, g = 5$ curve that lies on a hyperplane in X_8 could contribute to the degeneracy and they might account for the above discrepancy. We will leave this point to future investigation.

2.4. Degree 10 hypersurface in $\mathbf{WP}_{(5,2,1,1,1)}$

X_{10} is the hypersurface defined by a polynomial of homogeneous degree 10 in the weighted projective space $\mathbf{WP}_{(5,2,1,1,1)}$. It has $h^{1,1} = 1$, $h^{2,1} = 145$, $\chi = -288$, $c_2 = 34h$. The hyperplane section P has $6D = P \cdot P \cdot P = 1$, $c_2 \cdot P = 34$. Note that P is defined by a linear equation in x_3, x_4, x_5 only. The $(0,4)$ CFT has central charges

$$c_L = 6D + c_2 \cdot P = 35, \quad c_R = 6D + \tfrac{1}{2} c_2 \cdot P = 18.$$

A straightforward counting of D4-D0 bound state with D0 charge $0, 1$ determines the first two terms in the modified elliptic genus

$$Z_{X_{10}}(\tau, \bar{\tau}, y) = q^{-\frac{35}{24}} (3 + 2 \cdot (-288) q + \cdots) \theta_1(\bar{\tau}, y)$$

Requiring that $Z_{X_{10}}$ is a Jacobi form of weight $(-\frac{3}{2}, \frac{1}{2})$ then determines it to be

$$\begin{aligned} Z_{X_{10}}(\tau, \bar{\tau}, y) &= \eta(\tau)^{-35} \frac{541 E_4(\tau)^4 + 1187 E_4(\tau) E_6(\tau)^2}{576} \theta_1(\bar{\tau}, y) \\ &= q^{-\frac{35}{24}} (3 - 576q + 271704q^2 + 206401533q^3 + 21593767647q^4 + 1054723755951q^5 + \cdots) \theta_1(\bar{\tau}, y) \end{aligned} \tag{2.12}$$

A naive direct counting of D4 bound to 2 D0's give

$$(-288) \cdot (-289)/2 + (-288)\chi(\mathbf{P}^2) + 231200 = 271952$$

which is 248 more than the value 271704 predicted by modular invariance. Now a hyperplane through one point p_1 will also contain a whole curve with the same x_3, x_4, x_5 coordinates. There is again a correction to the degeneracy when p_2 lies on this curve, which is more subtle since the curve may degenerate depending on the x_3, x_4, x_5 coordinates. And furthermore, a genus g degree 1 curve C_g would have self-intersection $2g-3$ and the flux $C_g - J$ would carry D0 charge $2-g$. Such curves may contribute if they lie in a hyperplane section. A careful analysis of these contributions is beyond this note.

2.5. Bicubic in $\mathbf{P}^5$

$X_{3,3}$ is defined by

$$P_3(X) = Q_3(X) = 0 \tag{2.13}$$

in $\mathbf{P}^5$, where P and Q are generic cubic polynomials. $X_{3,3}$ has $h^{1,1} = 1$, $h^{2,1} = 73$, $\chi = -144$, $c_2 = 6h$. The hyperplane section P has $6D = P \cdot P \cdot P = 9$, $c_2 \cdot P = 54$. The $(0,4)$ CFT has

$$c_L = 6D + c_2 \cdot P = 63, \quad c_R = 6D + \tfrac{1}{2} c_2 \cdot P = 36.$$

The modified elliptic genus has the form

$$Z_{X_{3,3}}(\tau, \bar\tau, y) = \sum_{i=0}^{8} Z_i(\tau) \Theta_i^{(9)}(\bar\tau, y) \tag{2.14}$$

where $\Theta_i^{(m)}$ are defined as before. There is also the relation $Z_i = Z_{9-i}$. By direct counting from geometry we can determine the first few terms in the q-expansion of the Z_i's

$$\begin{aligned}
Z_0(\tau) &= q^{-\frac{63}{24}}(6 + 5 \cdot (-144)q + (?)q^2 + \cdots) \\
Z_1(\tau) &= q^{-\frac{77}{72}}(4 \cdot 1053 + 3 \cdot 1053 \cdot (-144+2)q + \cdots) \\
Z_2(\tau) &= q^{-\frac{29}{72}}(-3 \cdot 52812 + \cdots) \\
Z_3(\tau) &= q^{-\frac{5}{8}}(3 \cdot (-3402) + \cdots) \\
Z_4(\tau) &= q^{\frac{19}{72}}(2 \cdot 5520393 + \cdots)
\end{aligned} \tag{2.15}$$

We did not try to determine the (?) coefficient in $Z_0(\tau)$ because of the potential ambiguity in the counting from geometry. However we can count the first (non-polar) coefficient in

$Z_4(\tau)$ from degree 4 genus 1 curves in P, and together with the other polar coefficients they determine the modified elliptic genus completely.

The details of determining the modular forms are left to the appendix.[5] The first few terms in the q-expansion of the answer are given by

$$
\begin{aligned}
Z_0(\tau) &= q^{-\frac{63}{24}}(6 - 720q + 40032q^2 + 678474q^3 - 30885198768q^4 - 35708825468142q^5 \\
&\quad - 9448626104689554q^6 - 1170512868283650738q^7 - 88016808046791466314q^8 + \cdots) \\
Z_1(\tau) &= q^{-\frac{77}{72}}(4212 - 448578q - 374980104q^2 - 2020724648442q^3 - 890559631782378q^4 \\
&\quad - 147810582092632410q^5 - 13583665805416442478q^6 - 823655461162634305794q^7 + \cdots) \\
Z_2(\tau) &= q^{-\frac{29}{72}}(-158436 + 12471246q - 174600085086q^2 - 134299669045176q^3 \\
&\quad - 29070064587874050q^4 - 3172859337263652090q^5 - 218000892267121506858q^6 + \cdots) \\
Z_3(\tau) &= q^{-\frac{5}{8}}(-10206 + 13828428q - 24425287884q^2 - 35338801262184q^3 \\
&\quad - 9438086780879238q^4 - 1170314443959539166q^5 - 88014001223404540188q^6 + \cdots) \\
Z_4(\tau) &= q^{\frac{19}{72}}(11040786 - 6769752552q - 17629606262268q^2 - 5304774206609694q^3 \\
&\quad - 704390403350490336q^4 - 55554435778447164564q^5 + \cdots)
\end{aligned}
\tag{2.16}
$$

We can make one highly nontrivial check: the second coefficient in $Z_3(\tau)$, 13828428, is the number of D4 bound to 3 D2's with an extra D0-brane charge. From the geometric picture this comes from a degree 3 genus 0 curve $C_{3,0}$ lying in P, as well as a degree 3 genus 1 curve $C_{3,1}$ together with a pointlike instanton in P. The counting (using known results of Gopakumar-Vafa invariants that count $C_{3,0}$ and $C_{3,1}$) is

$$2 \cdot 6424326 + 2 \cdot (-3402) \cdot (-144) = 13828428$$

which precisely agrees with the prediction from modular invariance.

2.6. Quadriconic in $\mathbf{P}^7$

$X_{2,2,2,2}$ is defined by

$$P_2(X) = Q_2(X) = R_2(X) = S_2(X) = 0 \tag{2.17}$$

[5] As explained in the end of appendix A.2, this modular form has an unexpected feature, suggesting a yet uncovered mysterious relation among the polar coefficients.

in $\mathbf{P}^7$, where P, Q, R, S are generic quadratic polynomials. $X_{2,2,2,2}$ has $h^{1,1} = 1$, $h^{2,1} = 65$, $\chi = -128$, $c_2 = 4h$. The hyperplane section P has $6D = P \cdot P \cdot P = 16$, $c_2 \cdot P = 64$. The $(0, 4)$ CFT has

$$c_L = 6D + c_2 \cdot P = 80, \quad c_R = 6D + \tfrac{1}{2} c_2 \cdot P = 48.$$

The modular forms involved are more complicated and the determination of the D4-brane partition function in this case is left as a fun exercise for the reader.

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Appendix A. The details of modular forms

A.1. The method of generating modular representations

In this section we describe an algorithm to find many independent modular vectors to construct a basis of the relevant modular representation, the number of elements in the basis being the number of allowed polar terms in the q-expansion of the modular vector.

We can start with a vector $(\chi_i^w(\tau))_{i=0,\cdots,m-1}$ transforming in a particular m-dimensional modular representation with weight w (half integer in general), and obtain a weight $w + 2$ vector in the same representation by

$$\mathcal{D}_2(\chi_i^w)(\tau) := \frac{1}{2\pi i} \eta(\tau)^{2w} \partial_\tau (\eta(\tau)^{-2w} \chi_i^w(\tau)) \tag{A.1}$$

One can repeat this procedure and get modular vector of weight $w + 2n$. The modular vector obtained this way (for $n > 1$) are not necessarily the same as $\chi_i^w(\tau)$ multiplied by entire holomorphic modular forms (polynomials in E_4, E_6).

The first step is to find "seeding" modular forms $\chi_i^w(\tau)$ that transform in the same representation as $\Theta_{1,i}^m(\tau, y)$. Here the theta functions are defined as

$$\begin{aligned}
\Theta_{1,k}^m(\tau, y) &= \sum_{n \in \mathbf{Z} + \frac{1}{2} + \frac{k}{m}} (-)^{mn} q^{\frac{m}{2} n^2} z^{mn} \\
\Theta_{2,k}^m(\tau, y) &= \sum_{n \in \mathbf{Z} + \frac{1}{2} + \frac{k}{m}} q^{\frac{m}{2} n^2} z^{mn} \\
\Theta_{3,k}^m(\tau, y) &= \sum_{n \in \mathbf{Z} + \frac{k}{m}} q^{\frac{m}{2} n^2} z^{mn} \\
\Theta_{4,k}^m(\tau, y) &= \sum_{n \in \mathbf{Z} + \frac{k}{m}} (-)^{mn} q^{\frac{m}{2} n^2} z^{mn}
\end{aligned} \tag{A.2}$$

where $z = e^{2\pi i y}$, $k = 0, \cdots, m-1$. These are the usual Jacobi theta functions for $m = 1$, $k = 0$. Only the $\Theta^m_{1,k}(\tau, y)$'s form an m-dimensional modular representation by themselves. However they vanish at $y = 0$, and we need to come up with the $\chi^w_i(\tau)$'s that transform in the same way with some weight w.

A good set of seeding modular forms is

$$\begin{aligned}
\chi_i^{m,4l-\frac{m-1}{2}}(\tau) &= \theta_3(\tau)^{8l-m}\Theta^m_{3,i}(\tau) + \theta_4(\tau)^{8l-m}\Theta^m_{4,i}(\tau) + \theta_2(\tau)^{8l-m}\Theta^m_{2,i}(\tau), \qquad m \text{ odd;} \\
\chi_i^{m,4l-\frac{m-1}{2}}(\tau) &= \theta_3(\tau)^{8l-m}\Theta^m_{3,i}(\tau) + (-)^k\theta_4(\tau)^{8l-m}\Theta^m_{3,i}(\tau) + \theta_2(\tau)^{8l-m}\Theta^m_{2,i}(\tau), \quad m \text{ even.}
\end{aligned} \tag{A.3}$$

Here the first superscript of χ indicates its modular representation, i.e. that of Θ^m_1; the second superscript indicates its modular weight, and the subscript is the index for the modular vector. The choice of χ is motivated by the S and T transformation of the $\Theta^m_{i,k}$ of the form

$$\begin{aligned}
&\Theta^m_2 \xleftrightarrow{S} \Theta^m_3 \xleftrightarrow{T} \Theta^m_4, \qquad m \text{ odd} \\
&\Theta^m_2 \xleftrightarrow{S} \Theta^m_3 \xleftrightarrow{T} (-)^k\Theta^m_3, \qquad m \text{ even}
\end{aligned}$$

relative to the modular transform of Θ^m_1. There are in general more possible seeding modular forms, but (A.3) appears to suffice for our purpose.

A.2. The results

The modified elliptic genus of an M5-brane wrapped on the hyperplane section in the octic in $\mathbf{WP}_{4,1,1,1,1}$ is

$$\begin{aligned}
Z_{X_8}(\tau,\bar\tau,y) = \frac{1}{63}\eta^{-46}\sum_{k=0,1}&(77E_4^3E_6\chi_k^{2,\frac{7}{2}} - 19278E_6\Delta\chi_k^{2,\frac{7}{2}} - 168E_4^4\mathcal{D}_2\chi_k^{2,\frac{7}{2}} \\
&+ 245808E_4\Delta\mathcal{D}_2\chi_k^{2,\frac{7}{2}})\Theta^2_{1,k}(\bar\tau, y)
\end{aligned} \tag{A.4}$$

where $E_4, E_6, \Delta \equiv \eta^{24}$ and χ are understood to be functions of τ.

The modified elliptic genus of an M5-brane wrapped on the hyperplane section in the sextic in $\mathbf{WP}_{2,1,1,1,1}$ is

$$\begin{aligned}
Z_{X_6}(\tau,\bar\tau,y) = \frac{1}{4}\eta^{-45}\sum_{k=0}^{2}&(5E_4^3E_6\chi_k^{3,3} - 1344E_6\Delta\chi_k^{3,3} - 12E_4^4\mathcal{D}_2\chi_k^{3,3} \\
&+ 15360E_4\Delta\mathcal{D}_2\chi_k^{3,3})\Theta^3_{1,k}(\bar\tau, y)
\end{aligned} \tag{A.5}$$

The modified elliptic genus of an M5-brane wrapped on the hyperplane section in the bicubic in $\mathbf{P}^5$ is

$$
\begin{aligned}
Z_{X_{3,3}}(\tau,\bar\tau,y) = \frac{1}{698880}\eta^{-63}\sum_{k=0}^{8}\Big[&(-174720E_4^5E_6 - 39370048E_4^2E_6\Delta)\chi_k^{9,4}\\
&+(704340E_4^6 - 1205445441E_4^3\Delta + 143587676160\Delta^2)\mathcal{D}_2\chi_k^{9,4}\\
&+(176904E_4^4E_6 - 952935930E_4E_6\Delta)\mathcal{D}_2^2\chi_k^{9,4} + (-6368544E_4^5 + 2752749684E_4^2\Delta)\mathcal{D}_2^3\chi_k^{9,4}\\
&+(19105632E_4^3E_6 - 3794532480E_6\Delta)\mathcal{D}_2^4\chi_k^{9,4} + 7233791184E_4\Delta\mathcal{D}_2^5\chi_k^{9,4}\Big]\Theta_k^9(\bar\tau,y)
\end{aligned}
\tag{A.6}
$$

Here we constructed in fact one fewer basis modular vectors than all possible polar terms, nevertheless we seem to be lucky enough to match all the polar terms obtained from geometric counting. This suggests that there might be a hidden relation among the polar terms that is not determined by modular invariance.

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